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On survival probability of quantum states

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Abstract
The survival probability of a quantum state encodes essential information concerning the decay rate of quantum particles and is the primary object for investigating the time–energy uncertainty relations and the occurrence of the quantum Zeno effect. The purpose of this article is to uncover some curious properties concerning the relations between the values of the survival probability of a quantum state at different times. These relations put surprising restrictions on the evolution pictures of quantum states, and also illustrate their peculiar intricacies.

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The ways to manipulate the quantum states and to exploit the available energy resources to achieve the highest or desired decay rate (evolution speed) are closely related to deriving physical limits relevant to the quantum evolutions and to the theoretical issues in the newly emerging field of quantum computation and quantum information [6]. The purpose of this paper is to uncover some intrinsic and peculiar constraints on the evolution pictures of any quantum system specified by an initial state and a time-independent energy observable. These restrictions have their origin in the Fourier transform perspective of the notion of survival probability.

Consider the evolution of an arbitrary initial quantum state $|\psi\rangle$ (represented by a normalized wave function) driven by a time-independent energy observable (Hamiltonian) H, the evolving state $|\psi_t\rangle$ is determined by the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi_t\rangle = H|\psi_t\rangle, \qquad |\psi_0\rangle = |\psi\rangle,$$

where $\hbar$ is the Planck constant divided by 2π. Formally, the solution is given by $|\psi_t\rangle = \mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle$, and the survival probability at time t is defined as [7]

$$P_t = |\langle\psi|\psi_t\rangle|^2 = |\langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle|^2, \qquad t \in R.$$

Alternatively, the survival probability is the transition probability between the initial state $|\psi_0\rangle = |\psi\rangle$ and the final state $|\psi_t\rangle$.

The survival probability is a basic quantity in the study of the temporal behaviour of quantum mechanical systems. Mathematically, the survival probability of a quantum state can be expressed as the absolute square of the survival amplitude, which in turn can be expressed as the Fourier transform of the probability density induced by the wavefunction according to the Born rule in the energy representation. More explicitly, let $\{|E\rangle\}$ be the complete set of the energy eigenstates:

$$H|E\rangle = E|E\rangle, \qquad \langle E'|E\rangle = \delta(E' - E).$$

Let $|\psi\rangle$ be expanded in the energy eigenstates as

$$|\psi\rangle = \int \lambda(E)|E\rangle \mathrm{d}E,$$

where the integration (and also all subsequent integrations) is over the spectrum of H. When the energy spectrum is discrete, all integrals should be interpreted as discrete sums. Then

$$\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle = \int \mathrm{e}^{-\mathrm{i}tE/\hbar}\lambda(E)|E\rangle \mathrm{d}E,$$

and according to the Parseval theorem,

$$\langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle = \int \mathrm{e}^{-\mathrm{i}tE/\hbar}|\lambda(E)|^2 \mathrm{d}E.$$

Consequently, the survival amplitude (whose absolute square equals the survival probability) is precisely the Fourier transform of the state probability density $|\lambda(E)|^2$ in the energy representation. This intrinsic Fourier transform characteristics yield important implications for physical phenomena such as the time–energy uncertainty relations and the quantum Zeno effects [2–5, 8].

Now we proceed to present our main result. Specifically, we address the following problem:

Suppose that at a certain time $t = T$, we know the survival probability $P_T = \beta$, then what can we say about P_t at some other time, say $t = 2T$?

At first glance, one might claim that P_{2T} can be quite arbitrary, as long as we vary the quantum system (the initial quantum state and the energy observable) and keep $P_T = \beta$. Surprisingly, the following result establishes a rather stringent constraint for the possible values of P_{2T}.

Theorem. *Let $T > 0$ be a fixed instant of time, and suppose that $P_T = \beta$ (of course $0 \leqslant \beta \leqslant 1$). Then $\sqrt{P_{2T}} \geqslant 2\beta - 1$.*

Before giving the proof, let us elaborate a little on the singular intricacy of this result. From the theorem, we see that if $P_T > \frac{1}{2}$, then P_{2T} cannot be zero, no matter what the initial state and the energy observable are. Put it alternatively, if $P_{2T} = 0$, then it is necessary that $P_T \leqslant \frac{1}{2}$. The unique and subtle feature here is that we can find quantum systems such that $P_{2T} = 0$, and times t_1 and t_2 arbitrarily close to T such that $t_1 < T < t_2$ and $P_{t_1} = P_{t_2} = 1$ (see the following example). Nevertheless, we always have $P_T \leqslant \frac{1}{2}$. Of course, when $\beta \leqslant \frac{1}{2}$, the conclusion of the theorem is trivial. The meaningful part is the case $\beta > \frac{1}{2}$.

We may also formulate our motivating problem as follows: Suppose we know that $P_{2T} = 0$ for some fixed $T > 0$, then can we make any reasonable statement about P_t for $0 < t < 2T$? From the above theorem, we know that $P_T \leqslant \frac{1}{2}$, and moreover by iteration, $P_{T/2^n} \leqslant \cos^2(\pi/2^{n+2})$ for $n = 0, 1, 2, \ldots$. As for other t, the situations are radically different!

The factor $\frac{T}{2T} = \frac{1}{2}$ is really distinguished. It seems very hard to gain a physical intuition into this result which is of a number-theoretic flavour.

Let us consider an example, which shows that the inequality in the theorem is sharp, and also exhibits its peculiar features. Consider a two-level system with two energy eigenstates $|E_0\rangle$ and $|E_1\rangle$. For any non-negative integer $k = 0, 1, 2, \ldots,$ and any $T > 0$, let the energy observable be

$$H = \frac{(2k+1)\pi\hbar}{4T}(|E_0\rangle\langle E_0| - |E_1\rangle\langle E_1|)$$

and the initial state be

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|E_0\rangle + |E_1\rangle).$$

Then

$$P_t = \cos^2\left(\frac{(2k+1)\pi}{4T}t\right).$$

Therefore, $P_{2T} = 0$ and $P_T = \frac{1}{2}$, and the inequality $\sqrt{P_{2T}} \geqslant 2P_T - 1$ is saturated as an equality. This shows the optimality of our result. On the other hand, we have $P_t = 1$ when

$$t = \frac{2j}{2k+1}2T, \qquad j = 0, 1, \ldots, k.$$

Thus for fixed T, if we increase k, the points of t such that $P_t = 1$ become denser in the interval $[0, 2T]$, but P_T remains bounded by $\frac{1}{2}$. Actually, here $P_T = \frac{1}{2}$.

The mathematical content of the above theorem is known to probabilists (see [1], p 527). However, it is only stated as a problem there without proof, and it seems that the physical implications of this inequality are not illuminated in the quantum decaying context. We now present a proof of the theorem for completeness. Let Re and Im denote the real and imaginary parts of a complex number, respectively. For any real number $\gamma \in R$, we have

$$\begin{aligned}
(\mathrm{Re}(\mathrm{e}^{\mathrm{i}t\gamma/\hbar}\langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle))^2 &= \left(\mathrm{Re}\int \mathrm{e}^{\mathrm{i}t(\gamma-E)/\hbar}|\lambda(E)|^2\mathrm{d}E\right)^2 \\
&= \left(\int \cos(t(\gamma-E)/\hbar)|\lambda(E)|^2\mathrm{d}E\right)^2 \\
&\leqslant \int \cos^2(t(\gamma-E)/\hbar)|\lambda(E)|^2\mathrm{d}E \\
&= \int \frac{1}{2}(1+\cos(2t(\gamma-E)/\hbar))|\lambda(E)|^2\mathrm{d}E \\
&= \frac{1}{2}(1+\mathrm{Re}(\mathrm{e}^{\mathrm{i}2t\gamma/\hbar}\langle\psi|\mathrm{e}^{-\mathrm{i}2tH/\hbar}|\psi\rangle)) \\
&\leqslant \frac{1}{2}(1+\sqrt{P_{2t}}).
\end{aligned}$$

Therefore, we have

$$\max_{\gamma}(\mathrm{Re}(\mathrm{e}^{\mathrm{i}\gamma t}\langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle))^2 \leqslant \frac{1}{2}(1+\sqrt{P_{2t}}).$$

On the other hand,

$$\begin{aligned}
&(\mathrm{Re}(\mathrm{e}^{\mathrm{i}t\gamma/\hbar}\langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle))^2 \\
&\quad = (\cos(t\gamma/\hbar)\cdot\mathrm{Re}\langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle - \sin(t\gamma/\hbar)\cdot\mathrm{Im}\langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle)^2 \\
&\quad \leqslant (\mathrm{Re}\langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle)^2 + (\mathrm{Im}\langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle)^2 \\
&\quad = P_t,
\end{aligned}$$

and for any t, the above inequality can become an equality by choosing suitable γ, that is,

$$\max_\gamma(\mathrm{Re}(\mathrm{e}^{\mathrm{i}\gamma t}\langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle))^2 = P_t.$$

Consequently, we have

$$P_t \leqslant \tfrac{1}{2}(1+\sqrt{P_{2t}})$$

and the conclusion of the theorem follows.

Finally, let us formulate an extremal problem for the survival probability. For some fixed $T > 0$ and any time $t > 0$, let $M_t = \sup P_t$, the supremum is taken over all quantum systems (here a quantum system refers to a pair of an initial quantum state and a time-independent energy observable) whose survival probability at time $t = 2T$ is zero, that is, over all survival probabilities P_t such that $P_{2T} = 0$. From the above discussion, we know that for $t = \frac{T}{2^n}$,

$$M_t = \cos^2\left(\frac{\pi}{2^{n+2}}\right), \qquad n = 0, 1, \ldots$$

and $M_t = 1$ for any

$$t \in \left\{\frac{2j}{2k+1}2T : k = 1, 2, \ldots; j = 0, 1, 2, \ldots, k\right\}.$$

Note the latter set is dense in $[0, 2T]$.

More generally, for any positive, relatively prime integers m and n satisfying $2 \leqslant m < n$, put $d = \frac{n}{m}\frac{\pi}{T}\hbar$, and consider an m-level system with the energy observable

$$H = d\sum_{j=1}^{m} j|E_j\rangle\langle E_j|$$

and the initial state

$$|\psi\rangle = \frac{1}{\sqrt{m}}\sum_{j=1}^{m}|E_j\rangle.$$

Here $\{|E_j\rangle\}$ constitutes an orthonormal base for the system. Then the survival probability of $|\psi\rangle$ is

$$P_t = \left|\frac{1}{m}\sum_{j=1}^{m}\mathrm{e}^{-\mathrm{i}tjd/\hbar}\right|^2,$$

which simplifies to $P_t = \frac{1}{m^2}\left|\frac{\omega_t^m-1}{\omega_t-1}\right|^2$ if $\omega_t = \mathrm{e}^{-\mathrm{i}td/\hbar} = \mathrm{e}^{-\mathrm{i}2\pi\frac{n}{m}\frac{t}{2T}} \neq 1$, and to $P_t = 1$ if $\omega_t = 1$. Clearly, $P_{2T} = 0$ and $P_t = 1$ when t is an integer multiple of $\frac{m}{n}2T$. Also, we have $P_T = 0$ if n is even, and $P_T = 2\big(m^2\big(1-\cos\big(\frac{n}{m}\pi\big)\big)\big)^{-1}$ if n is odd.

Therefore, we conclude that for any t which is a rational multiple of $2T$ in $(0, 2T)$, not of the form $t = \frac{1}{n}2T$, we have $M_t = 1$. We conjecture that $M_t = 1$ for any t which is an irrational multiple of $2T$ in $(0, 2T)$, and $M_t = \cos^2\left(\frac{\pi}{2n}\right)$ when $t = \frac{1}{n}2T$ for some integer n.

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